2.2 The Inverse of a Matrix

Key idea: Some matrices admit multiplicative inverses (in the sense that if A $n\times n$, $A\cdot A^{-1} = A^{-1}\cdot A = I_n$). We will see that these matrices are particularly nice in that invertibility is tied to many notions we are already concerned with. For now, we concern ourselves with finding the inverse of a particular square matrix A.

We've already seen that the addition, subtraction, and multiplication we are used to extends to matrices. Today we investigate an extension of division.

Recall division in the real numbers: $5\cdot 5^{-1} = 5/5 = 1$ and $3\cdot 5^{-1} = 3/5 = 0.6$
Dividing by 5 is equivalent to multiplying by it's multiplicative inverse $5^{-1} = 1/5$.
(Similarly, subtraction by 5 is equivalent to adding by it's additive inverse -5.)
So, our notion of "division by a matrix A" will be multiplying by it's multiplicative inverse A^{-1}, i.e., the matrix s.t.

$$A\cdot A^{-1} = I_n = A^{-1}A$$

Here A, A^{-1} are $n\times n$ and both AA^{-1} and $A^{-1}A$ are important b/c commutativity fails for matrices.

This matrix is unique but more on that later.

Ex| Let $A = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}$, $B = \begin{bmatrix} 1/2 & 1/5 \\ -1 & -1/2 \end{bmatrix}$ and $C = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}$.

Show $C = A^{-1}$ and $B \neq A^{-1}$.

$$AC = \begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix} = CA.$$

✓ so $AC = I_2$, $CA = I_2$ so $C = A^{-1}$.

$$BA = \begin{bmatrix} 1/2 & 1/5 \\ -1 & -1/2 \end{bmatrix}\begin{bmatrix} 2 & 5 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} 0.8 & 2.1 \\ -1.5 & -4 \end{bmatrix}.$$

$BA \neq I_2$ so $B \neq A^{-1}$

So, we see inverting A is a subtle business, simply inverting all the elements of A failed to produce an inverse, which begs the question:

How do we find A^{-1} for a given A?

Fact: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$ then A is invertible and ($\leadsto$ has an inverse)

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

If $ad-bc = 0$, then A is not invertible $\leadsto$ has no inverse.

We call the quantity $ad-bc$ the determinant of A, written $\det(A)$,
if $\det(A) = 0$, A has no inverse and we sometimes call A a singular matrix,
if $\det(A) \neq 0$, A has an inverse, and we call it nonsingular.

Before describing why this is the inverse of a 2×2, we consider a familiar application.

Ex Let $A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$ and $\vec{b} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$. Find A^{-1} and solve $A\vec{x} = \vec{b}$.

Note $ad-bc = 3\cdot 6 - 4\cdot 5 = -2$ so $A^{-1} = \frac{1}{-2}\begin{bmatrix} 6 & -4 \\ -5 & 3 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}$.

Now, $A\vec{x} = \vec{b}$: $\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \end{bmatrix}$

$\Rightarrow A^{-1}A\vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 5/2 & -3/2 \end{bmatrix}\begin{bmatrix} 3 \\ 7 \end{bmatrix}$.

$\Rightarrow I_2\vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$

$\Rightarrow \vec{x} = A^{-1}\vec{b}$: $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$.

So matrix inverses allow the equation $A\vec{x} = \vec{b}$ to quickly be solved:

$$\vec{x} = A^{-1}\vec{b}.$$

Fact: If A is an invertible $n\times n$ matrix, then for each $\vec{b}$ in $\mathbb{R}^n$, the equation $A\vec{x} = \vec{b}$ has a unique solution: $\vec{x} = A^{-1}\vec{b}$.

So matrix inverses greatly help us in manipulating matrix equations so, we need understand how to find them in general.

Given an $n \times n$ matrix A, we seek $A^{-1} = [\vec{x}_1\ \vec{x}_2 \cdots \vec{x}_n]$ s.t. $AA^{-1} = I_n$.
By the definition of matrix multiplication, we see:

$$AA^{-1} = [A\vec{x}_1\ A\vec{x}_2 \cdots A\vec{x}_n] = [\vec{e}_1\ \vec{e}_2 \cdots \vec{e}_n] = I_n.$$

So we need only solve the n equations: $A\vec{x}_1 = \vec{e}_1$, $A\vec{x}_2 = \vec{e}_2 \cdots A\vec{x}_n = \vec{e}_n$.
And this can be done through row reduction!

Ex Find the inverse of $A = \begin{bmatrix} 1 & -2 & 1 \\ 4 & -7 & 3 \\ 2 & 6 & -5 \end{bmatrix}$.

We write $A^{-1} = [\vec{x}_1\ \vec{x}_2\ \vec{x}_3]$ and solve

$$A\vec{x} = \vec{e}_1 \quad \begin{bmatrix} 1 & 2 & 1 & 1 \\ 4 & -7 & 3 & 0 \\ 2 & 6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -17 \\ 0 & 1 & 0 & -14 \\ 0 & 0 & 1 & -10 \end{bmatrix} \Longrightarrow \vec{x}_1 = \begin{bmatrix} -17 \\ -14 \\ -10 \end{bmatrix}$$

$$A\vec{x} = \vec{e}_2 \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & -7 & 3 & 1 \\ 2 & 6 & 5 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 2 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$A\vec{x} = \vec{e}_3 \quad \begin{bmatrix} 1 & 2 & 1 & 0 \\ 4 & -7 & 3 & 0 \\ 2 & 6 & 5 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \Longrightarrow \vec{x} = \begin{bmatrix} -1 \\ -1 \\ -1 \end{bmatrix}$$

Thus $A^{-1} = \begin{bmatrix} -17 & 4 & -1 \\ -14 & 3 & -1 \\ -10 & 2 & -1 \end{bmatrix}$.

Notice A is row equivalent to I and in fact, could have solve all three equations at once because $[A\ I] \sim [I\ A^{-1}]$.

$$\text{Notice!!}\quad [A\ I] = \begin{bmatrix} 1 & 2 & 1 & 1 & 0 & 0 \\ 4 & -7 & 3 & 0 & 1 & 0 \\ 2 & 6 & 5 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & -17 & 4 & -1 \\ 0 & 1 & 0 & -14 & 3 & -1 \\ 0 & 0 & 1 & -10 & 2 & -1 \end{bmatrix} = [I\ A^{-1}].$$

You should check all the calculations mentioned above, including that indeed, $A^{-1} \cdot A = A \cdot A^{-1} = I_3$ and $[A\ I] \sim [I\ A^{-1}]$.

The algorithm for finding A^{-1}: If A is a square matrix and invertible then $A \sim I$ and $[A \; I] \sim [I \; A^{-1}]$. Else A is not invertible

This works for any matrix! Let's consider an example from the webpage: work / talk through that quick maple worksheet.

To finish a few facts:

1) If A is invertible, A^{-1} is invertible and $(A^{-1})^{-1} = A$.

2) If A and B are invertible, then AB is invertible and

$$(AB)^{-1} = B^{-1}A^{-1} \Rightarrow (AB)^{-1}(AB) = B^{-1}A^{-1}AB = B^{-1}IB = B^{-1}B = I.$$

3) If A is invertible, so is A^T and $(A^T)^{-1} = (A^{-1})^T$.

Part 2 generalizes to any product of invertible matrices: the inverse is the product of the inverses in reverse order,

$$(ABCD)^{-1} = D^{-1}C^{-1}B^{-1}A^{-1}.$$

Next time we consider a fundamental result in linear algebra: the consequences and characterizations of invertibility.